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RADIUS CONSTRAINTS AND MINIMAL EQUIPARTITION ENERGY OF RELATIVISTIC SYNCHROTRON SOURCES

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ABSTRACT

A measurement of the synchrotron self-absorption flux and frequency provides tight constraints on the physical size of the source and a robust lower limit on its energy. This lower limit is also a good estimate of the magnetic field and electrons' energy, if the two components are at equipartition. This well-known method was used for decades to study numerous astrophysical non-relativistic sources. Here we generalize the Newtonian equipartition theory to relativistic sources including the effect of deviation from spherical symmetry expected in such sources. We show that in the relativistic case a single epoch measurement of the synchrotron self-absorption flux and frequency is insufficient to constrain the radius and energy and at least two epochs, or knowledge of the time of the onset of the relativistic outflow, are needed. We show that like in the Newtonian case we can determine the emission radius and obtain a lower limit on the energy. We find that using the Newtonian formalism on a relativistic source would yield a smaller emission radius, and would generally yield a larger lower limit on the energy (within the observed region). For sources where the Synchrotron-self-Compton component can be identified, the minimization of the total energy is not necessary and we present an unambiguous solution for the parameters of the system.

Subject headings: radiation mechanisms: non thermal – methods: analytical

1. INTRODUCTION

The equipartition method (Pacholczyk 1970; Scott & Readhead 1977; Chevalier 1998) has been extensively applied to radio observations of Newtonian sources, in particular, to radio emission from supernovae (e.g., Shklovskii 1985, Slysh 1990, Chevalier 1998, Kulkarni et al. 1998, Li & Chevalier 1999, Chevalier & Fransson 2006, Soderberg et al. 2010a). The method relies on the fact that both the electron and magnetic field energy of a system, which emits self-absorbed synchrotron photons, depend sensitively on the source size. This allows for a robust determination of the size and of the minimal total energy needed to produce the observed emission. If the electron and magnetic field energies are close to equipartition then this lower limit is also a good estimate of their true energy. The strength of these arguments is that they are insensitive to the origin of the conditions within the emitting source and, as such, the results are independent of the details of the model.

The method depends only on the assumption of self absorbed synchrotron emission. In Newtonian sources it characterizes the emitting region with four unknowns. Three are microphysical: the number of electrons[3] that radiate in the observed frequency, their Lorentz Factor (LF) and the magnetic field. The macrophysical unknowns are the volume and area of the emitting region, which is assumed to be spherical and thus are both expressed by the fourth unknown: the source radius, R. An observed synchrotron spectrum, where the synchrotron self-absorption frequency is identified, provides three independent equations for the synchrotron frequency, the synchrotron flux and the blackbody flux. A fourth equation is needed to fully constrain the system. Luckily, as it turns out, the electron and magnetic energy depend sensitively on R in opposite ways and the total energy is minimized at some radius, in which the electrons and the magnetic field are roughly at equipartition. Thus the condition that the source energy is "reasonable" provides a robust estimate of R. We denote this radius, where the energy is minimal as R_{eq} and the corresponding minimal energy as E_{eq}. Thus, a single measurement of synchrotron self-absorption frequency, ν_a, and flux, $F_{\nu,a}$, provides a robust, almost model independent, estimate of the source size and its minimal energy.

An extension to the relativistic case is important, because of the existence of synchrotron sources that involve relativistic motion: jets in Gamma-Ray Bursts (GRB; e.g., Piran 2004), Active Galactic Nuclei (AGN; e.g., Krolik 1998), relativistic Type Ibc supernovae (e.g., Soderberg et al. 2010b); relativistic jets in tidal disruption event candidates (e.g., Zauderer et al. 2011) and others. Kumar & Narayan (2009) derived the constraints that synchrotron emission can put on a relativistic source in the context of the prompt optical and gamma-ray observations of GRB 080319B (the 'naked-eye burst'). This work was used later in the context of a tidal disruption event candidate (Zauderer et al. 2011).

Following the spirit of Kumar & Narayan (2009), we present here an explicit general extension of the Newtonian equipartition arguments to relativistic sources. This generalization introduces a new free parameter, the source's Lorentz factor, Γ. The solution requires an additional equation, which is the relation between R, Γ and the time

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[3] The calculations here are insensitive to the charge sign of the radiating particles, so if positrons are present then anywhere we refer to electrons we actually refer to pairs.

in the observer frame. Because of relativistic beaming geometrical effects[4] could be important[5], we include here the effect of deviation from spherical symmetry (both in Newtonian and relativistic outflows), which allows us to consider a general source geometry. We derive a general robust estimate of the radius of emission (Section 2). We derive the minimal total energy of the system, and discuss extra components that may lead to a larger energy (Section 3). We also present the constraints that arise when the self-absorption frequency is not identified but the radius of the source is directly measured, as may be the case for nearby objects (Section 4). Finally, we examine the case when the synchrotron-self-Compton component is observed and securely identified. In this case, minimization of the total energy is not necessary and all the parameters of the system can be solved unambiguously (Section 5). We summarize our results and consider some astrophysical implications in Section 6.

2. RADIUS ESTIMATE

Consider a source that produces synchrotron emission with an observed peak specific flux, $F_{\nu,p}$ at a frequency ν_p. The peak frequency will be given either by ν_a, the self-absorption frequency, or ν_m, the frequency at which electrons with the minimum LF are radiating, that is, $\nu_p = \max(\nu_a, \nu_m)$. We assume that ν_p is smaller than the cooling frequency, and thus we ignore the effect of electron cooling throughout this paper. We characterize the system by the following physical quantities: The total number of electrons within the observed region, N_e, the magnetic field in the source co-moving frame, B, the LF of the electrons that radiate at ν_p, γ_e, the size of the emitting region, R, and the LF of the source, Γ. We assume that the observed emission is dominated by the part of the source that moves within an angle of $1/\Gamma$ with respect to the line of sight. The observed synchrotron frequency is

$$\nu_p = \frac{eB\gamma_e^2\Gamma}{2\pi m_e c(1+z)}, \tag{1}$$

where e is the electron charge, m_e is the electron mass, c is the speed of light and z is the redshift. The observed synchrotron maximum specific flux, at ν_p, is

$$F_{\nu,p} = \frac{\sqrt{3}e^3 B N_e \Gamma^3 (1+z)}{\pi d_L^2 m_e c^2}, \tag{2}$$

where d_L is the luminosity distance[6]. The black-body specific flux, at frequency $\nu \leq \nu_a$, is given by

$$F_{\nu,BB} = \frac{2}{3}\nu^2(1+z)^3\Gamma m_e \gamma_e \frac{A}{d_L^2}, \tag{3}$$

where A is the area of the source we observe and we have used a black-body temperature $3kT \approx m_e c^2 \gamma_e$.

Using these equations, we can solve for the physical parameters of the system as a function of the observables and R and Γ. Using $\nu = \nu_a$ in eq. (3), the flux at this frequency is

$$F_{\nu_a,BB} = F_{\nu,p}\eta^{-\frac{1}{3}}, \tag{4}$$

where we have defined

$$\eta = \begin{cases} \frac{\nu_m}{\nu_a} & \text{if } \nu_a < \nu_m \\ 1 & \text{if } \nu_a > \nu_m. \end{cases} \tag{5}$$

We can now solve for three of the five unknowns:

$$\gamma_e = \frac{3F_{\nu,p}\eta^{\frac{5}{3}} d_L^2 \Gamma}{2\pi\nu_p^2(1+z)^3 m_e R^2 f_A} \approx 525\, F_{p,mJy}\, \nu_{p,10}^{-2}\, \eta^{\frac{5}{3}}\, (1+z)^{-3}\, d_{L,28}^2\, f_A^{-1}\, R_{17}^{-2}\, \Gamma, \tag{6}$$

$$N_e = \frac{9cF_{\nu,p}^3 \eta^{\frac{10}{3}} d_L^6}{8\sqrt{3}\pi^2 e^2 m_e^2 \nu_p^5 (1+z)^8 f_A^2 R^4} \approx 1\times 10^{54}\, F_{p,mJy}^3\, \nu_{p,10}^{-5}\, \eta^{\frac{10}{3}}\, (1+z)^{-8}\, d_{L,28}^6\, f_A^{-2}\, R_{17}^{-4}, \tag{7}$$

$$B = \frac{8\pi^3 m_e^3 c \nu_p^5 (1+z)^7 R^4 f_A^2}{9e\Gamma^3 F_{\nu,p}^2 \eta^{\frac{10}{3}} d_L^4} \approx (1.3\times 10^{-2}\,\mathrm{G})\, F_{p,mJy}^{-2}\, \nu_{p,10}^5\, \eta^{-\frac{10}{3}}\, (1+z)^7\, d_{L,28}^{-4}\, f_A^2\, R_{17}^4\, \Gamma^{-3}, \tag{8}$$

where $F_{p,mJy} = F_{\nu,p}/\mathrm{mJy}$ and, throughout the paper, we use the usual notation $Q_n = Q/10^n$ in cgs units. To account for non-spherical geometry we have introduced $f_A = A/(\pi R^2/\Gamma^2) \leq 1$, the area filling factor, and $f_V = V/(\pi R^3/\Gamma^4) \leq 1$, the volume filling factor that will be used shortly.

[4] Note that since the true geometrical parameters that affect the observations are the volume and area, the commonly used Newtonian formalism relays on the assumption of spherical symmetry, without explicitly deriving the possible effects of deviations from that symmetry on the results.

[5] We consider sources that move along (or close enough to) the line of sight, otherwise the radiation will be beamed away from us.

[6] The observed region taken here is up to an angle $1/\Gamma$ with respect to the line of sight. Note that this is different than eq. (5) on Sari, Piran & Narayan (1998; see, also, Kumar & Narayan 2009), where they take $F_\nu \propto \Gamma$, since they take an average of F_ν over the entire sphere, while we consider electrons that are only within an angle of $1/\Gamma$.

A given observer detects emission from an angle $1/\Gamma$ with respect to the line of sight. The total energy in electrons within this observed region is

$$E_e = N_e m_e c^2 \gamma_e \Gamma = \frac{27 c^3 F_{\nu,p}^4 d_L^8 \eta^5 \Gamma^2}{16\sqrt{3}\pi^3 e^2 m_e^2 \nu_p^7 (1+z)^{11} f_A^3 R^6}$$
$$\approx (4.4\times 10^{50}\mathrm{erg})\, F_{p,mJy}^4\, \nu_{p,10}^{-7}\, \eta^5\, (1+z)^{-11}\, d_{L,28}^8\, f_A^{-3}\, R_{17}^{-6}\, \Gamma^2, \tag{9}$$

while the total energy in the magnetic field is

$$E_B = \frac{(B\Gamma)^2}{8\pi} V = \frac{8\pi^6 m_e^6 c^2 \nu_p^{10} (1+z)^{14} R^{11} f_A^4 f_V}{81 e^2 F_{\nu,p}^4 d_L^8 \eta^{\frac{20}{3}} \Gamma^8}$$
$$\approx (2.1\times 10^{46}\mathrm{erg})\, F_{p,mJy}^{-4}\, \nu_{p,10}^{10}\, \eta^{-\frac{20}{3}}\, (1+z)^{14}\, d_{L,28}^{-8}\, f_A^4\, f_V\, R_{17}^{11}\, \Gamma^{-8}, \tag{10}$$

where we have taken the width of the emitting region to be $\sim R/\Gamma^2$. Note that in the case of a continuous outflow, where the flow is wider than $\sim R/\Gamma^2$, the flow is not in causal contact and in this case this formalism applies only to the emission of a particular region (or "blob") that dominates the observed emission.

The Newtonian (spherically symmetric) equipartition radius and minimal total energy are

$$R_N \approx (1.7\times 10^{17}\mathrm{cm})\, F_{p,mJy}^{\frac{8}{17}}\, \nu_{p,10}^{-1}\, \eta^{\frac{35}{51}}\, (1+z)^{-\frac{25}{17}}\, d_{L,28}^{\frac{16}{17}},$$
$$E_N \approx (2.5\times 10^{49}\mathrm{erg})\, F_{p,mJy}^{\frac{20}{17}}\, \nu_{p,10}^{-1}\, \eta^{\frac{15}{17}}\, (1+z)^{-\frac{37}{17}}\, d_{L,28}^{\frac{40}{17}}. \tag{11}$$

Using R_N and E_N we can express the total energy in the general case as:

$$E = E_e + E_B = E_N \left(\frac{f_V^{\frac{6}{17}}}{f_A^{\frac{9}{17}} \Gamma^{\frac{26}{17}}}\right) \left[\frac{11}{17}\left(\frac{R}{R_{eq}}\right)^{-6} + \frac{6}{17}\left(\frac{R}{R_{eq}}\right)^{11}\right], \tag{12}$$

where R_{eq} (which equals R_N in the spherical Newtonian case) is

$$R_{eq} = R_N\, f_A^{-\frac{7}{17}}\, f_V^{-\frac{1}{17}}\, \Gamma^{\frac{10}{17}} \approx (1.7\times 10^{17}\mathrm{cm})\, F_{p,mJy}^{\frac{8}{17}}\, \nu_{p,10}^{-1}\, \eta^{\frac{35}{51}}\, (1+z)^{-\frac{25}{17}}\, d_{L,28}^{\frac{16}{17}}\, f_A^{-\frac{7}{17}}\, f_V^{-\frac{1}{17}}\, \Gamma^{\frac{10}{17}}. \tag{13}$$

The total energy is minimized with respect to R at R_{eq}, with $E_B \approx (6/11)E_e$. Eq. (13) includes the effect of deviation from spherical symmetry, showing that R_{eq} is insensitive to the volume filling factor, f_V, and weakly sensitive to the area filling factor f_A. A considerable deviation from spherical symmetry is required to affect the radius estimate. With $\Gamma = 1$, eq. (13) provides the geometrical corrections to the Newtonian radius. A high value of Γ significantly increases R_{eq}, thus, using the Newtonian formalism for an ultrarelativistic source will result in a significant underestimate of its radius. Also, by including the parameter η, eq. (13) takes into account the more general possibility of having the peak of the spectrum at ν_m, whereas it is commonly taken to be at ν_a. We note that, since the total energy is a very strong function of radius, then R_{eq} provides a robust estimate of R, unless we allow the energy to be significantly higher than the minimal total energy we will derive in the next section.

In the relativistic case, the total energy, eq. (12) depends on two unknowns: R and Γ. An attempt to look for a global minimum of the energy with respect to both R and Γ fails as it leads to two incompatible equations. We therefore minimized the energy with respect to R. We need now another relation that will enable us to express Γ as a function of R. To obtain this relation we introduce an extra observable, t, the time, in the observer frame, since the onset of the relativistic outflow. In most astrophysical scenarios Γ evolves on a time scale comparable to, or longer than, t and:

$$t \approx \frac{R(1-\beta)(1+z)}{\beta c}, \tag{14}$$

where β is the velocity of the outflow at observer time t. If the time of the onset of the outflow is known then a single measurement of the synchrotron spectrum is enough and eqs. (13) and (14) are solved simultaneously[7] to determine R and Γ. In the extreme relativistic limit $\Gamma \gg 1$, $t \approx R(1+z)/(2c\Gamma^2)$, and

$$R_{eq} \approx (3.4\times 10^{17}\mathrm{cm})\, F_{p,mJy}^{\frac{2}{3}}\, \nu_{p,10}^{-\frac{17}{12}}\, \eta^{\frac{35}{36}}\, (1+z)^{-\frac{5}{3}}\, d_{L,28}^{\frac{4}{3}}\, f_A^{-\frac{7}{12}}\, f_V^{-\frac{1}{12}}\, t_d^{-\frac{5}{12}}, \tag{15}$$

where t_d is the time measured in days.

If the onset of the outflow is unknown, then we need at least two epochs, t_1 and t_2, at which $F_{\nu,p}$ and ν_p (and ν_a if it is not the peak frequency) are measured. If $\Gamma(t_1) \sim \Gamma(t_2)$, we solve eqs. (13) and (14) for $R(t_2)$ and $R(t_1)$ and t_2 and t_1, assuming $\Gamma(t_1) = \Gamma(t_2)$. However, Γ may evolve on a time scale comparable to t. Therefore if $t_2 \gg t_1$ it is possible that $\Gamma(t_1) \nsim \Gamma(t_2)$. This case is identified if the above procedure results in $R(t_2) \gg R(t_1)$. Then $t_2 - t_1 \sim t$ and $R(t_2) - R(t_1) \sim R$, and we can approximate the solution at t_2 using $t_2 \approx t$. In this case the solution of $R(t_1)$ cannot be trusted.

[7] Strictly speaking, in this case we have to substitute $\Gamma(R)$ from eq. (14) in eqs. (9) and (10) and minimize the total energy with respect to R. However, it can be shown that this procedure yields almost identical results to solving eqs. (13) and (14) simultaneously.

3. THE MINIMAL (EQUIPARTITION) ENERGY

We turn now to various constraints that can be put on the total source energy. We begin with the absolute lower limit and continue examining other components that can contribute to the source energy.

3.1. *An absolute lower limit*

The absolute minimal total energy of the system accounts only for the electrons that radiate at ν_p and for the magnetic field in the observed region, namely within an angle of $1/\Gamma$ from the line of sight:

$$E_{eq} = E_N\, f_A^{-\frac{9}{17}}\, f_V^{\frac{6}{17}}\, \Gamma^{-\frac{26}{17}} \approx (2.5\times 10^{49}\mathrm{erg})\, F_{p,mJy}^{\frac{20}{17}}\, \nu_{p,10}^{-1}\, \eta^{\frac{15}{17}}\, (1+z)^{-\frac{37}{17}}\, d_{L,28}^{\frac{40}{17}}\, f_A^{-\frac{9}{17}}\, f_V^{\frac{6}{17}}\, \Gamma^{-\frac{26}{17}}. \tag{16}$$

This lower limit decreases with Γ, and, therefore, it is less stringent for relativistic sources. This is driven mostly by the increased beaming, and thus the reduced area and volume within an angle of $\sim 1/\Gamma$.

The effect of deviation from spherical geometry is opposite for f_A and f_V. For a very narrow jet with half-opening angle θ_j smaller than $1/\Gamma$, $f_A = f_V = (\theta_j\Gamma)^2$, we find $R_{eq} \propto \theta_j^{-16/17}\Gamma^{-6/17}$, see eq. (13), and the minimal energy is $E_{eq} \propto \theta_j^{-6/17}\Gamma^{-32/17}$. If θ_j decreases below $1/\Gamma$, the resulting minimal energy will be larger than the one in eq. (16) with $f_A = f_V = 1$, and it will exceed the Newtonian equipartition energy for $\theta_j \lesssim 1/\Gamma^{16/3}$. Thus, a jet narrower than $1/\Gamma$ requires larger energy to produce the observed emission. The reason for this effect is not trivial (as there are competing effects), but the main driver is the reduction in the area, which increases γ_e, eq. (6), and also reduces B. Both lead to a significant increase of R_{eq} and to a higher E_{eq}.

When $\theta_j > 1/\Gamma$, the resulting minimal energy will be larger than the one in eq. (16) with $f_A = f_V = 1$. The jet in this case has more energy that we do not observe directly since we only observe a region within $1/\Gamma$. Thus, the minimal lower limit (and thus the most robust one) is obtained by assuming $\theta_j = 1/\Gamma$.

This last situation is one of several cases in which additional energy is "hidden" in the system and is not observed directly, but it influences, of course, the overall energy budget. We now turn to examine additional energy reservoirs of this kind. These reservoirs obviously always increase the overall energy of the system.

3.2. *Wide outflows*

Eq. (16) with $f_A = f_V = 1$ corresponds to the minimal energy within an observed region of angle $\sim 1/\Gamma$ with respect to the line of sight. This is the minimal energy required to produce the observed signal. However, the overall energy of the source would be larger, as briefly discussed above, if the outflow's half opening angle is larger than $1/\Gamma$. In this case the flow will carry an energy larger than the calculated so far in eq. (16), with $f_A = f_V = 1$, by a factor of $4\Gamma^2(1-\cos\theta_j)$. This additional energy is not observed directly and the "true" energy can be determined only if an independent estimate of the jet opening angle is available (such as in GRBs, when a "jet break" takes place and θ_j can be estimated, e.g., Sari, Piran & Halpern 1999). The resulting relativistic equipartition energy estimate is larger than the Newtonian one for $\theta_j \gtrsim 1/(\sqrt{2}\Gamma^{4/17})$.

3.3. *Electrons that radiate at ν_m*

Above we considered only the electrons that radiate at ν_p. These electrons are likely to carry most of the relativistic electron energy if $\nu_p = \nu_m$. However, if $\nu_m < \nu_a$ most of the electrons' energy is carried by the electrons with the minimal Lorentz factor, γ_m (and whose emission is self absorbed). In this case the electrons' energy will be larger than that of eq. (9) (with $\eta = 1$), by a factor of $\left(\frac{\gamma_m}{\gamma_e}\right)^{2-p}$, where p is the electron energy distribution power-law and $p > 2$. In rare cases ν_m can be identified in the spectrum by a transition from $F_\nu \propto \nu^2$ to $F_\nu \propto \nu^{5/2}$. In these cases $\left(\frac{\gamma_m}{\gamma_e}\right)^{2-p} = \left(\frac{\nu_m}{\nu_a}\right)^{\frac{2-p}{2}}$, so the radius estimate is hardly modified, since it is only multiplied by $\left(\frac{\nu_m}{\nu_a}\right)^{\frac{2-p}{34}}$ in eq. (13) (with $\eta = 1$). The total minimal energy is somewhat increased, since it is multiplied by $\left(\frac{\nu_m}{\nu_a}\right)^{\frac{11(2-p)}{34}}$ in eq. (16) (with $\eta = 1$). In the most common case where ν_m is not measured it must be evaluated theoretically. This can be done if the electrons are known to be accelerated by a shock with LF similar to that of the source, Γ. In that case $\gamma_m = \chi_e(\Gamma-1)$ where $\chi_e = \frac{p-2}{p-1}\epsilon_e\frac{m_p}{m_e}$ and ϵ_e is the fraction of the protons energy that goes into electrons and m_p is the proton mass (if γ_m is found to be $\gamma_m < 2$, then one should use $\gamma_m = 2$). With this, and following the same procedure as above of setting $E_B \approx (6/11)E_e$, the radius where the energy is minimal becomes

$$\begin{aligned} R_{eq} \approx &(1\times 10^{17}\mathrm{cm})\,[21.8(525)^{p-1}]^{\frac{1}{13+2p}}\,\chi_e^{\frac{2-p}{13+2p}}\,F_{p,mJy}^{\frac{6+p}{13+2p}}\,\nu_{p,10}^{-1} \\ &\times(1+z)^{-\frac{19+3p}{13+2p}}\,d_{L,28}^{\frac{2(p+6)}{13+2p}}\,f_A^{-\frac{5+p}{13+2p}}\,f_V^{-\frac{1}{13+2p}}\,\Gamma^{\frac{p+8}{13+2p}}\,(\Gamma-1)^{\frac{2-p}{13+2p}}. \end{aligned} \tag{17}$$

The corresponding minimal total energy within $\sim 1/\Gamma$ is:

$$E_{eq} \approx (1.3\times 10^{48}\mathrm{erg})\,[21.8]^{-\frac{2(p+1)}{13+2p}}\left[(525)^{p-1}\chi_e^{2-p}\right]^{\frac{11}{13+2p}} F_{p,mJy}^{\frac{14+3p}{13+2p}}\,\nu_{p,10}^{-1}$$
$$\times (1+z)^{-\frac{27+5p}{13+2p}}\,d_{L,28}^{\frac{2(3p+14)}{13+2p}}\,f_A^{-\frac{3(p+1)}{13+2p}}\,f_V^{\frac{2(p+1)}{13+2p}}\,\Gamma^{-\frac{5p+16}{13+2p}}\,(\Gamma-1)^{-\frac{11(p-2)}{13+2p}}. \quad (18)$$

The last two expressions reduce to eqs. (13) and (16) with $\eta = 1$ for $p = 2$ (when all electrons carry a similar amount of energy). For the Newtonian case, $\gamma_m = 2$ and $p = 3$, one obtains the solution found by Chevalier (1998).

3.4. *Hot protons*

If the source contains protons it is reasonable to expect that these take a significant share of the total internal and bulk energy. For example, observations indicate that in shock heated gas (for example, in GRB afterglows, see, e.g., Panaitescu & Kumar 2002) most of the energy is carried by hot protons. The exact fraction of the total energy carried by other components is unknown, but these observations suggest that the fraction carried by electrons, ϵ_e, is typically ~ 0.1 in relativistic shocks and lower in Newtonian shocks. Using this parameterization, the energy carried by the hot protons is $E_p \approx E_e/\epsilon_e$. This implies a total matter energy of $E_e + E_p = \xi E_e$, where $\xi \equiv 1+\epsilon_e^{-1}$. Similarly, the parameters at which the energy is minimal are found by setting $E_B \approx (6/11)\xi E_e$. The radius estimate is hardly modified, since it is only multiplied by $\xi^{\frac{1}{17}}$, $\xi^{\frac{1}{12}}$ and $\xi^{\frac{1}{13+2p}}$ in eqs. (13), (15) and (17), respectively. The total minimal energy is somewhat increased, since it is multiplied by $\xi^{\frac{11}{17}}$ and $\xi^{\frac{11}{13+2p}}$ in eqs. (16) and (18), respectively.

4. SYSTEMS WITH MEASURED R BUT UNKNOWN ν_A

There are cases, especially for Galactic and local Universe sources, in which we can resolve and measure the source's size on the sky and determine $R\psi = \theta_{obs} d_A$, where $\psi \equiv \min(1/\Gamma, \theta_j)$, θ_{obs} is the half-angular extent of the source and $d_A = d_L(1+z)^{-2}$ is the angular distance. However, for these sources we do not always have a measurement of ν_a. We can still estimate a minimal total energy carried by the magnetic field and by electrons that radiate at the observed frequency ν at a flux F_ν. This was first done in the Newtonian case by Burbidge (1959; see, e.g., Nakar, Piran & Sari 2005, for a recent example) and the relativistic case, without considering any geometrical factors, was discussed in Dermer & Atoyan (2004; see also Dermer & Menon 2009). Determining the LF of electrons radiating at ν with (1) and the number of radiating electrons within $1/\Gamma$ with (2), we can determine the total energy of the system. It is minimized once $E_B \approx (3/4)E_e$, which yields an equipartition magnetic field (see, e.g., Dermer & Menon 2009)

$$B_{eq} \approx (5\times 10^{-3}\mathrm{G})\,F_{\nu,mJy}^{\frac{2}{7}}\,\nu_{10}^{\frac{1}{7}}\,(1+z)^{\frac{11}{7}}\left(\frac{\theta_{obs}}{10\mathrm{mas}}\right)^{-\frac{6}{7}}\left(\frac{d_L}{10\mathrm{kpc}}\right)^{-\frac{2}{7}} f_V^{-\frac{2}{7}}\,\Gamma^{-\frac{1}{7}}\,\psi^{\frac{6}{7}}, \quad (19)$$

and a total minimal energy within $\sim 1/\Gamma$ of

$$E_{eq} \approx (2.8\times 10^{40}\mathrm{erg})\,F_{\nu,mJy}^{\frac{4}{7}}\,\nu_{10}^{\frac{2}{7}}\,(1+z)^{-\frac{20}{7}}\left(\frac{\theta_{obs}}{10\mathrm{mas}}\right)^{\frac{9}{7}}\left(\frac{d_L}{10\mathrm{kpc}}\right)^{\frac{17}{7}} f_V^{\frac{3}{7}}\,\Gamma^{-\frac{16}{7}}\,\psi^{-\frac{9}{7}}, \quad (20)$$

where $F_{\nu,mJy} = F_\nu/mJy$. For these nearby sources we usually have the time of the onset of the outflow and can estimate Γ. This allows us to use (20) to estimate the absolute minimum total energy.

5. SYNCHROTRON-SELF-COMPTON EMISSION

If synchrotron-self-Compton (SSC) emission is also observed, then there is no need to minimize the total energy. This introduces two additional observables that allow us to determine all parameters of the system without the need of minimizing the total energy. This was done by Chevalier & Fransson (2006) and later by Katz (2012) for the Newtonian case. Here, we extend these estimates to the relativistic case, again, following the spirit of Kumar & Narayan (2009; see also Dermer & Atoyan 2004). If the synchrotron emission peaks in the radio band and the SSC observed emission is in the X-rays, then it is safe to assume, as we do in the following, that the Klein-Nishina effects can be neglected.

The SSC peak frequency, ν_p^{SSC}, and the ratio of the synchrotron to the SSC luminosities are

$$\nu_p^{SSC} \approx \nu_p \gamma_e^2, \quad (21)$$

and

$$\frac{B^2/8\pi}{U_{ph}} \approx \frac{\nu_p F_{\nu,p}}{\nu_p^{SSC} F_{\nu,p}^{SSC}}, \quad (22)$$

where U_{ph} is the photon energy density (in the co-moving frame) and $F_{\nu,p}^{SSC}$ is the SSC peak flux. The photon energy density can be approximated as $U_{ph} = \frac{\nu_p F_{\nu,p}}{\Gamma^2 c}\frac{d_L^2}{R^2}$.

Consider an observed SSC frequency ν, such that $\nu_p^{SSC} < \nu$, with observed flux F_ν. These observations are related to the peak of the SSC component as

$$F_\nu = F_{\nu,p}^{SSC}\left(\frac{\nu}{\nu_p^{SSC}}\right)^{-\frac{p-1}{2}}. \quad (23)$$

Using eqs. (21), (22), (23), (6) and (8) we can solve for the radius of emission

$$R \approx (1\times 10^{17}\mathrm{cm})\,[5(525)^{p-3}]^{\frac{1}{2(2+p)}}\,F_{p,mJy}^{\frac{1}{2}}\,\nu_{p,10}^{-\frac{3+2p}{2(2+p)}}\,\eta^{\frac{5(1+p)}{6(2+p)}}$$
$$\times(1+z)^{-\frac{5+3p}{2(2+p)}}\,d_{L,28}\,f_A^{-\frac{1+p}{2(2+p)}}\,\Gamma^{\frac{1+p}{2(2+p)}}\,\left(\frac{F_{\nu,p}}{F_\nu}\right)^{\frac{1}{2(2+p)}}\,\left(\frac{\nu_p}{\nu}\right)^{\frac{p-1}{4(2+p)}}. \tag{24}$$

This expression and eq. (14) allow us to determine the radius of emission and Γ of the source. We can then substitute the obtained values for R and Γ in eqs. (6)-(10) and obtain all physical parameters of the emitting region. In the extreme relativistic limit $\Gamma \gg 1$, we can solve for all these parameters analytically (see Appendix A for these expressions).

Finally, we note that for $\Gamma = 1$ and $p = 3$, eq. (24) reduces to the radius estimate in Katz (2012) for the Newtonian case, within a factor of ~ 2. This small discrepancy appears simply because our expression for the synchrotron frequency, eq. (1), is larger than the one used by Katz (2012) by this same factor.

6. SUMMARY

We have extended the equipartition arguments of Newtonian synchrotron sources in spherical geometry to include relativistic sources in general geometry. This enables to derive robust estimates of the radius and of the minimal total energy of the emitting region of a large variety of synchrotron transient sources. It also enables to quantify the effect of the, typically unknown, geometry on the robustness of these estimates.

We find that in the relativistic case the estimate of the emission radius is increased by a factor of $\Gamma^{10/17}$ compared with the Newtonian case. The lower limit on the energy (within a region of $\sim 1/\Gamma$) is lower by $\Gamma^{-26/17}$ compared with the Newtonian one. Therefore, using the Newtonian formalism for a relativistic source underestimates (overestimates) the emission radius (lower limit on the energy). We show that in order to find if relativistic corrections are needed, and to estimate Γ, at least two epochs of measurements are needed, or alternatively the time since the onset of the relativistic outflow should be known.

The collimation of relativistic sources affects the energy lower limit. It is minimal for a source with $\theta_j \sim 1/\Gamma$ and it increases for either narrower or wider jets. Thus, the lower limit obtained when $\theta_j \sim 1/\Gamma$ is assumed is the most robust one. A wider jet involves additional energy that we do not observe directly as it is beamed elsewhere, while the reason why a narrower jet also requires more energy is less trivial and is discussed above. For an emitting region smaller [larger] than $\sim 1/\Gamma$, the lower limit on the energy is larger than that for an emitting region of $\sim 1/\Gamma$ by a factor of $(\theta_j\Gamma)^{-6/17}$ [$\sim 2(\theta_j\Gamma)^2$] and approaches the Newtonian lower limit only for extremely small [very large] values of θ_j.

The energy estimates discussed above involve the minimal energy (of the electrons and the magnetic field) required to produce the observed radiation. However, additional components in which energy is "hidden" may exist in the system. These include: 1. The extra energy carried by electrons with minimal Lorentz Factor γ_m and whose synchrotron frequency ν_m is self-absorbed, such that $\nu_m < \nu_a$, and 2. the energy carried by protons, if they are present in the source. We consider their possible effect on the total energy required. We find that these extra sources of energy hardly change the emission radius, while the total minimal energy is increased.

Finally, we extend the Newtonian equipartition formalism to relativistic sources in two other scenarios. First, for nearby sources, where we are able to identify the angular size of the source on the sky, but the self absorption frequency is not identified. Second, for when a synchrotron-self-Compton component is identified, in addition to the synchrotron self-absorption, and there are two additional observables that enable us to directly determine all parameters of the emitting region. Overall we find that relativistic corrections can be important and that using the Newtonian formula for a relativistic source would lead to significantly inaccurate results.

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APPENDIX A

If a reliable measurement of the SSC flux is available, then there is no need to minimize the total energy; all parameters of the emitting region can be uniquely determined (see Section 5). In the extreme relativistic limit $\Gamma \gg 1$, eq. (14) is $t \approx R(1+z)/(2c\Gamma^2)$, and we can solve for all these parameters analytically as follows. The radius of emission will be given by eq. (24) as

$$R \approx (1\times 10^{17}\mathrm{cm})\,C_1^{\frac{2}{7+3p}}\,C_2^{\frac{2(1+p)}{7+3p}}\,F_{p,mJy}^{\frac{2(2+p)}{7+3p}}\,\nu_{p,10}^{-\frac{2(3+2p)}{7+3p}}\,\eta^{\frac{10(1+p)}{3(7+3p)}}$$
$$\times(1+z)^{-\frac{9+5p}{7+3p}}\,d_{L,28}^{\frac{4(2+p)}{7+3p}}\,f_A^{-\frac{2(1+p)}{7+3p}}\,t_d^{-\frac{1+p}{7+3p}}\,\left(\frac{F_{\nu,p}}{F_\nu}\right)^{\frac{2}{7+3p}}\,\left(\frac{\nu_p}{\nu}\right)^{\frac{p-1}{7+3p}}, \tag{A-1}$$

and Γ will be given by

$$\Gamma \approx C_1^{\frac{1}{7+3p}}\, C_2^{\frac{4(2+p)}{7+3p}}\, F_{p,mJy}^{\frac{2+p}{7+3p}}\, \nu_{p,10}^{-\frac{3+2p}{7+3p}}\, \eta^{\frac{5(1+p)}{3(7+3p)}} \times (1+z)^{-\frac{1+p}{7+3p}}\, d_{L,28}^{\frac{2(2+p)}{7+3p}}\, f_A^{-\frac{1+p}{7+3p}}\, t_d^{-\frac{2(2+p)}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{\frac{1}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{\frac{p-1}{2(7+3p)}}, \tag{A-2}$$

where $C_1 \approx 5(525)^{p-3}$ and $C_2 \approx 4.4$. With these two expressions, the rest of the parameters can be determined by substituting them in eqs. (6)-(10) as follows:

$$\gamma_e \approx 525 C_1^{-\frac{3}{7+3p}}\, C_2^{\frac{4}{7+3p}}\, F_{p,mJy}^{\frac{1}{7+3p}}\, \nu_{p,10}^{-\frac{5}{7+3p}}\, \eta^{\frac{20}{3(7+3p)}} \times (1+z)^{-\frac{4}{7+3p}}\, d_{L,28}^{\frac{2}{7+3p}}\, f_A^{-\frac{4}{7+3p}}\, t_d^{-\frac{2}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{-\frac{3}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{\frac{3(1-p)}{2(7+3p)}}, \tag{A-3}$$

$$N_e \approx 1\times 10^{54}\, C_1^{-\frac{8}{7+3p}}\, C_2^{-\frac{8(1+p)}{7+3p}}\, F_{p,mJy}^{\frac{5+p}{7+3p}}\, \nu_{p,10}^{-\frac{11-p}{7+3p}}\, \eta^{\frac{10(3-p)}{3(7+3p)}} \times (1+z)^{-\frac{4(5+p)}{7+3p}}\, d_{L,28}^{\frac{2(5+p)}{7+3p}}\, f_A^{-\frac{2(3-p)}{7+3p}}\, t_d^{\frac{4(1+p)}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{-\frac{8}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{\frac{4(1-p)}{7+3p}}, \tag{A-4}$$

$$B \approx (1.3\times 10^{-2}\,\mathrm{G})\, C_1^{\frac{5}{7+3p}}\, C_2^{-\frac{4(4+p)}{7+3p}}\, F_{p,mJy}^{-\frac{4+p}{7+3p}}\, \nu_{p,10}^{\frac{5(4+p)}{7+3p}}\, \eta^{-\frac{5(9+p)}{3(7+3p)}} \times (1+z)^{\frac{4(4+p)}{7+3p}}\, d_{L,28}^{-\frac{2(4+p)}{7+3p}}\, f_A^{\frac{9+p}{7+3p}}\, t_d^{\frac{2(4+p)}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{\frac{5}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{-\frac{5(1-p)}{2(7+3p)}}, \tag{A-5}$$

$$E_e \approx (4.4\times 10^{50}\,\mathrm{erg})\, C_1^{-\frac{10}{7+3p}}\, C_2^{\frac{4(1-p)}{7+3p}}\, F_{p,mJy}^{\frac{2(4+p)}{7+3p}}\, \nu_{p,10}^{-\frac{19+p}{7+3p}}\, \eta^{\frac{5(11-p)}{3(7+3p)}} \times (1+z)^{-\frac{5(5+p)}{7+3p}}\, d_{L,28}^{\frac{4(4+p)}{7+3p}}\, f_A^{-\frac{11-p}{7+3p}}\, t_d^{-\frac{2(1-p)}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{-\frac{10}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{\frac{5(1-p)}{7+3p}}, \tag{A-6}$$

$$E_B \approx (2.1\times 10^{46}\,\mathrm{erg})\, C_1^{\frac{14}{7+3p}}\, C_2^{-\frac{2(21+5p)}{7+3p}}\, F_{p,mJy}^{\frac{2p}{7+3p}}\, \nu_{p,10}^{\frac{2(14+p)}{7+3p}}\, \eta^{-\frac{10(7-p)}{3(7+3p)}} \times (1+z)^{\frac{7-5p}{7+3p}}\, d_{L,28}^{\frac{4p}{7+3p}}\, f_A^{\frac{2(7-p)}{7+3p}}\, f_V\, t_d^{\frac{21+5p}{7+3p}} \left(\frac{F_{\nu,p}}{F_\nu}\right)^{\frac{14}{7+3p}} \left(\frac{\nu_p}{\nu}\right)^{-\frac{7(1-p)}{7+3p}}. \tag{A-7}$$

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